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Padé approximants and Eisenstein–Ramanujan continued fraction

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Abstract

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A corresponding type continued fraction expansion for Eisenstein–Ramanujan series is considered and certain contraction forms of this continued fraction are provided. The relevant convergents are realized as direct Padé approximants and their block structure in the Padé table is studied. Further, it is shown that the resulting sequence of convergents provides an efficient tool for analyzing the natural boundary of the above series. A new interpretation is given to those convergents which are as such not any of the Padé approximants, but in fact they are actually identified as partial Padé approximants in view of their order of contact with the original series.

Keywords: Padé approximants; Padé table; continued fractions.

1. Introduction

Continued fractions and Padé approximants have been the subject of much recent interest in many fields of applications. There is a close connection between these two, since the convergents of certain types of continued fractions happen to be Padé approximants. For full details of continued fractions, their correspondence properties and the relationship with Padé approximants, see, for example, [3,4,6,7,10,14]. However, for the sake of completeness we shall first state some fundamentals.

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The *normal* $[m/n]$ Padé approximant of the formal power series

$$f(z)=\sum_{i=0}^{\infty} c_i z^i, \quad c_0 \neq 0, \tag{1.1}$$

is a rational function

$$\frac{P_{m,n}(z)}{Q_{m,n}(z)}=\frac{p_0+p_1 z+\cdots+p_m z^m}{1+q_1 z+\cdots+q_n z^n}, \quad m, n \geqslant 0, \tag{1.2}$$

such that

$$f(z) Q_{m,n}(z)-P_{m,n}(z)=\mathrm{O}(z^{m+n+1}), \tag{1.3}$$

where $\mathrm{O}(z^k)$ denotes a power series in ascending powers of z beginning with a term in z^k.

We denote a continued fraction C by

$$C=\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots, \tag{1.4}$$

where a_n and b_n are functions of a (complex) variable z. Let

$$C_n=\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots+\frac{a_n|}{|b_n}=\frac{P_n}{Q_n}$$

be its nth convergent. Both P_n and Q_n satisfy the recurrence relation

$$U_n=b_n U_{n-1}+a_n U_{n-2}, \quad n=2,3, \ldots, \tag{1.5}$$

with $P_0=0$, $P_1=a_1$, $Q_0=1$, $Q_1=b_1$.

This paper is devoted to the study of the Eisenstein–Ramanujan continued fraction, which has the following form:

$$E(z)=\frac{1|}{|1}-\frac{z|}{|1}-\frac{z^2-z|}{|1}-\frac{z^3|}{|1}-\frac{z^4-z^2|}{|1}-\cdots. \tag{1.6}$$

Here, $a_{2n}=-z^{2n-1}$, $a_{2n+1}=-(z^{2n}-z^n)=-z^n(z^n-1)$, $b_n=1$.

In [8], it has been shown that for $|z|<1$, this continued fraction equals the continued product

$$\prod_{n=1}^{\infty}(1+z^n)(1-z^{2n})=\sum_{n=0}^{\infty} z^{n(n+1)/2}. \tag{1.7}$$

The general form of (1.6) was given by Ramanujan [16]. He suggested the identity

$$\begin{aligned} & 1-az+a^2 z^3-a^3 z^6+a^4 z^{10}-\cdots \\ & \quad=\frac{1|}{|1}+\frac{az|}{|1}+\frac{a(z^2-z)|}{|1}+\frac{az^3|}{|1}+\frac{a(z^4-z^2)|}{|1}+\cdots. \end{aligned} \tag{1.8}$$

This, in fact, is a special case of a general continued fraction of Ramanujan, namely

$$R(a, b, \Omega, z)=\frac{1|}{|1}+\frac{az+\Omega z|}{|1}+\frac{bz+\Omega z^2|}{|1}+\frac{az^2+\Omega z^3|}{|1}+\frac{bz^2+\Omega z^4|}{|1}+\cdots, \tag{1.9}$$

which has been discussed in detail in [2,11]. In [illegible] number of special cases of (1.9) is considered, among which (1.6). Here we study the continued fraction (1.6) and its power series expansion (1.7) from the point of view of Padé approximants.

The first four convergents of (1.6) are

$$\frac{1}{1}, \quad \frac{1}{1-z}, \quad \frac{1+z-z^2}{1-z^2}, \quad \frac{1+z-z^2-z^3}{1-z^2-z^3+z^4}.$$

From the degrees of the fraction and from the power series expansion it follows that

$$\frac{1}{1} = 1 + O(z) = [0/0], \qquad \frac{1}{1-z} = 1 + z + O(z^2) = [0/1],$$

$$\frac{1+z-z^2}{1-z^2} = 1 + z + z^3 + z^5 + O(z^7) = [2/2].$$

For the fourth convergent, there is a simplification possible:

$$\frac{1+z-z^2-z^3}{1-z^2-z^3+z^4} = \frac{1+2z+z^2}{1+z-z^3} = 1 + z + z^3 + z^6 + O(z^7) = [2/3].$$

This occurs in all the convergents for $n \geqslant 4$, because

$$\frac{P_{2n+3}}{Q_{2n+3}} = \cdots + \frac{z^{2n} - z^n |}{| 1} - \frac{z^{2n+1} |}{| 1} - \frac{z^{2n+2} - z^{n+1} |}{| 1}$$

$$= \cdots + \frac{z^n(z^n - 1)(1 - z^{2n+2} + z^{n+1})}{(1 - z^{2n+2}) + z^{n+1}(1 - z^n)},$$

$$\frac{P_{2n+4}}{Q_{2n+4}} = \cdots + \frac{z^{2n+2} - z^{n+1} |}{| 1} - \frac{z^{2n+3} |}{| 1} = \cdots + \frac{z^{n+1}(z^{n+1} - 1)}{1 - z^{2n+3}}.$$

It is thus natural for us to try to expand $E(z)$ into a corresponding type continued fraction (C-fraction) so that its convergents are generally Padé approximants. Moreover, one of the important properties of the C-fraction happens to be that its convergents are irreducible [18, p.331].

Section 2 deals with the expansion of $E(z)$ into a C-fraction and an immediate identification of its convergents as Padé approximants. Nearly one-third of the convergents as such are not Padé approximants according to the customary definition, but they are only partial Padé approximants in the sense that their order of contact with the original series is less than that of normal approximants. Some of the contraction forms of the derived C-fraction are examined in Section 3 and the natural boundary of $E(z)$ is considered in Section 4. The block structure of convergents in the Padé table is studied in Section 5.

2. C-fraction expansion of $E(z)$

The C-fractions were first introduced by Leighton and Scott [15]. These authors proved that every power series can have a unique C-fraction expansion. Scott and Wall [18] considered the relationship of C-fractions with the Padé table. They called the C-fraction and its power series *regular* in case all of the convergents qualify to be Padé approximants. Frank [9] further enriched this kind of investigations and gave an elegant algorithm for expanding an arbitrary power series into a C-fraction. She also reformulated the earlier results for a class of regular C-fractions. The notion of α-regularity was introduced to characterize the regular C-fractions in terms of geometrical properties of the Padé table.

Using the algorithm given in [9] we may expand $E(z)$ into a C-fraction:

$$E(z)=\frac{1|}{|1}-\frac{z|}{|1}+\frac{z|}{|1}+\frac{z^1|}{|1}-\frac{z|}{|1}+\frac{z|}{|1}+\frac{z^2|}{|1}-\frac{z|}{|1}+\frac{z|}{|1}+\frac{z^3|}{|1}-\cdots. \tag{2.1}$$

In view of the relation (1.5), the computed convergents of (2.1) are

$$\begin{aligned}&\frac{1}{1},\quad \frac{1}{1-z},\quad \frac{1+z}{1},\quad \frac{1+2z}{1+z-z^2},\quad \frac{1+z-z^2}{1-z^2},\\ &\frac{1+2z+z^2}{1+z-z^3},\quad \frac{1+2z+2z^2+z^3-z^4}{1+z+z^2-z^3-z^4},\quad \frac{1+z-z^4}{1-z^3},\ldots .\end{aligned} \tag{2.2}$$

The denominators of the convergents are given explicitly by

$$Q_1=1,\qquad Q_2=1-z,\qquad Q_3=1,$$

$$\left.\begin{aligned}&Q_{3n+1}=\sum_{i=0}^{n}z^i-\sum_{i=1}^{n}z^{n+i},\\ &Q_{3n+2}=1-z^{n+1},\\ &Q_{3n+3}=\sum_{j=0}^{n}z^j-\sum_{j=1}^{n}z^{n+1+j},\end{aligned}\right\}\quad n=1,2,3,\ldots . \tag{2.3}$$

Also it can be shown that

$$\left.\begin{aligned}&\deg[P_{3n+1}]=\deg[P_{3n+2}]=\tfrac{1}{2}(n^2+n+2),\\ &\deg[P_{3(n+1)}]=\tfrac{1}{2}(n^2+n+4),\end{aligned}\right.\quad n=2,3,\ldots . \tag{2.4}$$

From the degrees of the convergents and from the expansions, it follows that

$$\frac{P_1}{Q_1}=\frac{P_{0,0}}{Q_{0,0}}=\frac{1}{1}=1+O(z)=[0/0],$$

$$\frac{P_2}{Q_2}=\frac{P_{0,1}}{Q_{0,1}}=\frac{1}{1-z}=1+z+O(z^2)=[0/1],$$

$$\frac{P_3}{Q_3}=\frac{P_{1,0}}{Q_{1,0}}=\frac{1+z}{1}=1+z+O(z^2)=[1/0],$$

$$\frac{P_4}{Q_4}=\frac{P_{1,2}}{Q_{1,2}}=\frac{1+2z}{1+z-z^2}=1+z+z^3+O(z^4)=[1/2],$$

$$\frac{P_5}{Q_5}=\frac{P_{2,2}}{Q_{2,2}}=\frac{1+z-z^2}{1-z^2}=1+z+z^3+O(z^5)=[2/2],$$

$$\frac{P_6}{Q_6}=\frac{P_{2,3}}{Q_{2,3}}=\frac{1+2z+z^2}{1+z-z^3}=1+z+z^3+z^6+O(z^7)=[2/3],$$

$$\frac{P_7}{Q_7}=\frac{P_{4,4}}{Q_{4,4}}=\frac{1+2z+2z^2+z^3-z^4}{1+z+z^2-z^3-z^4}=1+z+z^3+z^6+O(z^8):$$

no Padé approximant,

$$\vdots$$

$$\frac{P_{10}}{Q_{10}}=\frac{P_{7,6}}{Q_{7,6}}=\frac{1+2z+2z^2+3z^3+z^4-z^5-z^7}{1+z+z^2+z^3-z^4-z^5-z^6}=1+z+z^3+z^6+z^{10}+O(z^{13}):$$

no Padé approximant,

$$\vdots$$

In general, from the power series expansions, it can be verified that

$$\frac{P_{3i+1}}{Q_{3i+1}}=\frac{P_{m,n}}{Q_{m,n}}=1+z+z^3+z^6+\cdots+z^{(i+1)(i+2)/2}+O(z^{m+n}),\quad i=2,3,4,\ldots:$$

no Padé approximant, (2.5)

where

$$2m=i^2+i+2,\quad n=2i,\quad \text{by (2.3) and (2.4).}$$

We observe that the convergents given by (2.2) are not as such all Padé approximants. For example, the 7th, 10th, 13th,... order convergents are not Padé approximants. Though strictly not Padé fractions, they are still responsible for generating the subsequent convergents which are really Padé approximants. Therefore, there is sufficient reason to believe that these convergents ought to be Padé fractions though under certain weak conditions. In view of (1.3) and (2.5), each of these convergents satisfies the identity

$$E(z)Q_{m,n}(z)-P_{m,n}(z)=O(z^{m+n}). \tag{2.6}$$

To be normal Padé approximants, each convergent should fit one more term of the series. Therefore, these rational functions may be termed as *Almost Padé Approximants* (APAs). The continued fraction (2.1) is a rare example in the class of C-fractions because non-Padé convergents occur periodically (see (2.5)). Identifying (2.1) with the most general C-fraction

$$f(z)=\frac{a_1z^{\alpha_1}|}{|1}+\frac{a_2z^{\alpha_2}|}{|1}+\frac{a_3z^{\alpha_3}|}{|1}+\cdots, \tag{2.7}$$

we see that each of its convergents satisfies the identities

$$P_n(z)Q_{n+1}(z)-P_{n+1}(z)Q_n(z)=(-1)^{n+1}a_1a_2\cdots a_{n+1}z^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}, \tag{2.8}$$

$$E(z)Q_n(z)-P_n(z)=(-1)^n a_1a_2\cdots a_{n+1}z^{\alpha_1+\alpha_2+\cdots+\alpha_{n+1}}+\cdots. \tag{2.9}$$

Let p_n and q_n be the degrees of $P_n(z)$ and $Q_n(z)$, respectively. If $P_n(z)/Q_n(z)$ is a normal Padé fraction, then

$$\alpha_1+\alpha_2+\cdots+\alpha_{n+1}>p_n+q_n. \tag{2.10}$$

In the case of non-Padé convergents considered here, the above inequality becomes an equality by virtue of (2.6). Let

$$P_n(z)=\sum_{i=0}^{p}a_iz^i,\qquad Q_n(z)=\sum_{j=0}^{q}b_jz^j,\qquad E(z)=\sum_{k=0}^{\infty}c_kz^k,$$

$$R_{m,n}=\begin{bmatrix} c_n & c_{n-1} & \cdots & c_{n-m}\\ c_{n+1} & c_n & \cdots & c_{n-m+1}\\ \vdots & & & \\ c_{n+m-1} & c_{n+m-2} & \cdots & c_{n-1}\end{bmatrix},\quad c_k=0,\quad \text{for } k<0,$$

$$P=\begin{bmatrix} a_0\\ a_1\\ \vdots\\ a_p\\ 0\\ \vdots\end{bmatrix},\qquad Q=\begin{bmatrix} b_0\\ b_1\\ \vdots\\ b_q\\ 0\\ \vdots\end{bmatrix}. \tag{2.11}$$

Equation (1.3) can then be expressed [9, p.93] in the form

$$R_{q,p+1}Q=0,\qquad R_{p+1,0}Q=P. \tag{2.12}$$

In the case of APAs we see that the first of the above relations is not satisfied. The exact positions of convergents in the Padé table and their block formations are as shown in Fig. 1. A few of the interesting aspects of these are discussed in Section 5.

3. Some explicit contraction forms of continued fractions

We have observed that the 7th, 10th, 13th,... order convergents of (2.1) are not really Padé approximants. Let us contract (2.1) in such a way that all its convergents are perfectly Padé fractions. So our aim is now to construct a new continued fraction, called the *two-third part* of (2.1), in order to exclude the 1st, 4th, 7th, 10th, 13th,... order convergents of it. For this purpose, to start with, we take the general continued fraction

$$C=b_0+\frac{a_1|}{|b_1}+\frac{a_2|}{|b_2}+\frac{a_3|}{|b_3}+\cdots, \tag{3.1}$$

whose nth convergent P_n/Q_n is given by

$$\begin{bmatrix} P_n\\ Q_n\end{bmatrix}=\begin{bmatrix} P_{n-1} & P_{n-2}\\ Q_{n-1} & Q_{n-2}\end{bmatrix}\begin{bmatrix} b_n\\ a_n\end{bmatrix},\quad n=1,2,3,\ldots, \tag{3.2}$$

where $P_{-1}=1$, $Q_{-1}=0$, $P_0=b_0$, $Q_0=1$. Let $T_0, T_1, T_2,\ldots$ be the convergents of the two-third

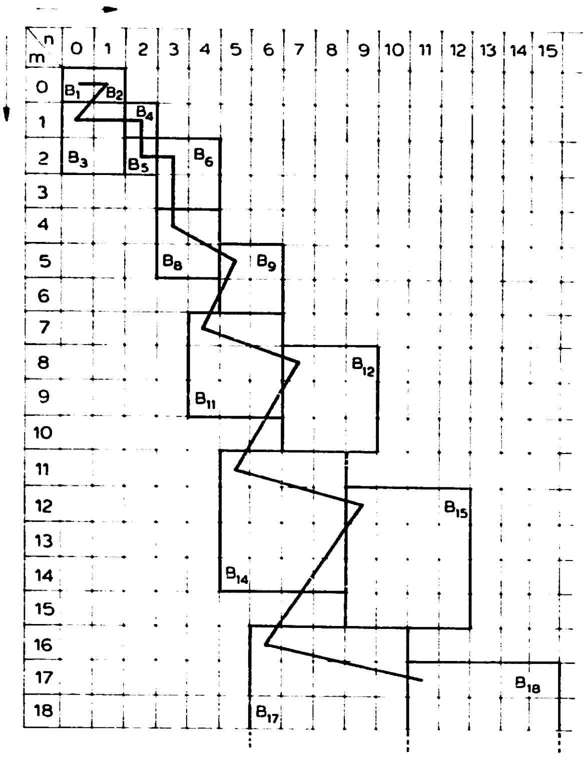

Fig. 1. Block structure for convergents of the C-fraction (2.1).

part of (3.1). The elements of the sequence $\{P_{3n+1}/Q_{3n+1}\}_0^\infty$ are not members of the required two-third part. Then clearly

$$T_{2n-1} = \frac{P_{3n-1}}{Q_{3n-1}}, \quad T_{2n} = \frac{P_{3n}}{Q_{3n}}, \qquad n = 0, 1, 2, \dots . \tag{3.3}$$

By (3.2) we have

$$P_{3n} = b_{3n} P_{3n-1} + a_{3n} P_{3n-2}, \tag{3.4a}$$

$$P_{3n-1} = b_{3n-1} P_{3n-2} + a_{3n-1} P_{3n-3}, \tag{3.4b}$$

$$P_{3n-2} = b_{3n-2} P_{3n-3} + a_{3n-2} P_{3n-4}. \tag{3.4c}$$

Eliminating P_{3n-2} from (3.4) we arrive at

$$P_{3n-1} = (b_{3n-1} b_{3n-2} + a_{3n-1}) P_{3n-3} + b_{3n-1} a_{3n-2} P_{3n-4}, \tag{3.5a}$$

$$b_{3n-1} P_{3n} = (b_{3n} b_{3n-1} + a_{3n}) P_{3n-1} - a_{3n} a_{3n-1} P_{3n-3}. \tag{3.5b}$$

Similar expressions hold for the denominators Q also. The relationships (3.5) involving P and similar two relations involving Q connect the numerators and denominators of three successive convergents of the contracted continued fraction. From (3.5a) we observe that $b_{3n-2} b_{3n-1} + a_{3n-1}$ and $a_{3n-2} b_{3n-1}$ are, respectively, the $(2n-1)$th partial denominator and numerator of the contracted continued fraction. Similarly (3.5b) determines the $2n$th partial denominator and numerator. Since $P_0/Q_0 = b_0$, the partial numerators of the required continued fraction are

$$a_1b_2, \quad -a_2a_3/b_2, \quad a_4b_5, \quad -a_5a_6/b_5, \ldots,$$

and the partial denominators are

$$b_1b_2+a_2, \quad (b_2b_3+a_3)/b_2, \quad b_4b_5+a_5, \quad (b_5b_6+a_6)/b_5, \ldots .$$

Thus, after an equivalence transformation we reach the two-third part of (3.1) as

$$\begin{aligned} &b_0+\frac{a_1b_2\,|}{|(b_1b_2+a_2)}-\frac{a_2a_3\,|}{|(b_2b_3+a_3)}+\frac{a_4b_2b_5\,|}{|(b_4b_5+a_5)}-\frac{a_5a_6\,|}{|(b_5b_6+a_6)} \\ &\quad+\cdots+\frac{a_{3n-2}b_{3n-4}b_{3n-1}\,|}{|(b_{3n-2}b_{3n-1}+a_{3n-1})}-\frac{a_{3n-1}a_{3n}\,|}{|(b_{3n-1}b_{3n}+a_{3n})}+\cdots. \end{aligned} \tag{3.6}$$

It is obvious that there are two other two-third parts of (3.1). Here we present them for the sake of completeness. Proceeding along similar lines we derive, as above, the two-third part of (3.1) lacking in the sequence (P_{3n}/Q_{3n}) as

$$\begin{aligned} &\frac{b_0b_1+a_1}{b_1}-\frac{a_1a_2/b_1\,|}{|(b_1b_2+a_2)}+\frac{a_3b_1b_4\,|}{|(b_3b_4+a_4)}-\frac{a_4a_5\,|}{|(b_4b_5+a_5)} \\ &\quad+\cdots-\frac{a_{3n-1}a_{3n-2}\,|}{|(b_{3n-2}b_{3n-1}+a_{3n-1})}+\frac{a_{3n}b_{3n-2}b_{3n+1}\,|}{|(b_{3n}b_{3n+1}a_{3n+1})}-\cdots. \end{aligned} \tag{3.7}$$

The two-third part of (3.1) lacking on the sequence $\{P_{3n+2}/Q_{3n+2}\}_0^\infty$ is

$$\begin{aligned} &b_0+\frac{a_1|}{|b_1}+\frac{a_2b_3\,|}{|(b_2b_3+a_3)}-\frac{a_3a_4\,|}{|(b_3b_4+a_4)}+\frac{a_5b_3b_6\,|}{|(b_5b_6+a_6)}-\frac{a_6a_7\,|}{|(b_6b_7+a_7)} \\ &\quad+\cdots+\frac{a_{3n-1}b_{3n-3}b_{3n}\,|}{|(b_{3n-1}b_{3n}+a_{3n})}-\frac{a_{3n}a_{3n+1}\,|}{|(b_{3n}b_{3n+1}+a_{3n+1})}+\cdots. \end{aligned} \tag{3.7a}$$

Then, using in (2.1) the formula (3.6) derived above, we obtain

$$E(z)=\frac{1\,|}{|1-z}+\frac{z^2\,|}{|1+z}+\frac{z\,|}{|1-z}+\frac{z^2\,|}{|1+z}+\cdots. \tag{3.8}$$

The nature of construction of (3.8) ensures that all the convergents are Padé approximants. Applying the formulae (3.7) and (3.7a) we get two other two-third parts of (2.1) as

$$E(z)=1+\frac{z\,|}{|1-z}+\frac{z\,|}{|1+z}+\frac{z^2\,|}{|1-z}+\frac{z\,|}{|1+z^2}+\frac{z^3\,|}{|1-z}+\cdots \tag{3.9}$$

and

$$E(z)=\frac{1|}{|1}-\frac{z\,|}{|1+z}-\frac{z^2\,|}{|1+z}-\frac{z\,|}{|1+z}-\frac{z^3\,|}{|1+z^2}-\frac{z\,|}{|1+z}-\cdots. \tag{3.10}$$

Each of the above contracted forms of (2.1) has some interesting regularity structure built into the elements so that the general term can be written down rather quickly. We can contract further the above set of continued fractions. Applying the even part and odd part contraction formulae given in [14] to each of the forms (3.8)–(3.10), we get another set of contracted forms:

$$E(z)=\frac{1+z|}{|1}-\frac{z^3|}{|1+z}-\frac{z^4|}{|1+z^2}-\frac{z^5|}{|1+z^3}-\frac{z^6|}{|1+z^4}-\cdots, \tag{3.11}$$

$$E(z)=\frac{1}{1-z}\left[1-\frac{z^2|}{|1+z}-\frac{z^3|}{|1+z^2}-\frac{z^4|}{|1+z^3}-\cdots\right], \tag{3.12a}$$

$$E(z)=\frac{1|}{|1-z}+\frac{z^2(1-z)|}{|1+z(1-z)}-\frac{z^3|}{|1+z^2}-\frac{z^4|}{|1+z^3}-\frac{z^5|}{|1+z^4}-\cdots. \tag{3.12b}$$

The other set of contracted forms is omitted here because they do accommodate non-Padé convergents also. In fact the continued fractions (3.12a) and (3.12b) are equivalent in the sense that they give rise to the same sequence of convergents. The computed convergents of (3.11) and (3.12) are given in the next section. They are, respectively, $C_{3(n+1)}$ and C_{3n+2} of (2.2), that is, (3.11) and (3.12) are the "one-third" of (2.2).

4. Natural boundary of $E(z)$

The convergents of (3.12) provide information on the natural boundary of $E(z)$. We recall, if every point on the circle of convergence of a given power series is a singular point, then the circle of convergence of that series is its own natural boundary. If, in the power series

$$f(z)=\sum a_n z^n, \quad a_n=0, \tag{4.1}$$

except when n belongs to a sequence n_k such that

$$n_{k+1}>(1+\epsilon)n_k, \quad \epsilon>0, \tag{4.2}$$

then the circle of convergence of the series is a natural boundary of the function [19, Theorem 7.43, p.223]. In the case of $E(z)$ we have

$$n_k=\tfrac{1}{2}k(k+1), \quad \text{so that } \frac{n_{k+1}}{n_k}=1+\frac{2}{k}.$$

As $k\to\infty$, $2/k\to 0$. Therefore, the gaps in $E(z)$ are not large enough to be Hadamard gaps. The Hadamard theorem cannot be applied here. However, the convergents of (3.12) give indication that $|z|=1$ is the natural boundary of $E(z)$. The convergents of (3.12) are

$$\frac{1}{1-z},\quad \frac{1+z-z^2}{1-z^2},\quad \frac{1+z-z^4}{1-z^3},\quad \frac{1+z+z^3-z^4-z^5+z^6-z^7}{1-z^4},\ \ldots .$$

In general, its nth convergent $f_n(z)=P_n(z)/Q_n(z)=\hat{P}_n(z)/(1-z)\hat{Q}_n(z)$ is given by

$$\hat{P}_n(z)=(1+z^{n-1})\hat{P}_{n-1}(z)-z^n\hat{P}_{n-2}(z), \quad n=2,3,\ldots, \tag{4.3}$$

with $\hat{P}_0=\hat{P}_1=1$ and $\hat{Q}_n(z)=1+z+z^2+\cdots+z^{n-1}$, $\hat{Q}_0=0$, $\hat{Q}_1=1$, by recurrence (4.3) and by the induction method. Hence,

$$Q_{n(z)}=1-z^n, \quad n=1,2,\ldots . \tag{4.4}$$

We can seek the expansions of $f_n(z)$ in the following form:

$$f_1(z) = 1 + z + z^2 + z^3 + \cdots,$$
$$f_2(z) = 1 + z + z^3 + z^5 + z^7 + z^9 + \cdots,$$
$$f_3(z) = 1 + z + z^3 + z^6 + z^9 + z^{12} + z^{15} + \cdots,$$
$$f_4(z) = 1 + z + z^3 + z^6 + z^{10} + z^{14} + z^{18} + z^{22} + \cdots,$$
$$f_5(z) = 1 + z + z^3 + z^6 + z^{10} + z^{15} + z^{20} + z^{25} + \cdots,$$
$$\vdots$$

In general,

$$f_n(z) = 1 + z + z^3 + z^6 + \cdots + z^{n(n+1)/2} + z^{n(n+3)/2} + z^{n(n+5)/2} + \cdots,$$

which can be proved by induction. This implies that

$$|E(z) - f_n(z)| = |z^{(n+1)(n+2)/2} + \cdots - z^{n(n+3)/2} - z^{n(n+5)/2} - \cdots| \leqslant \sum_{i=n(n+1)/2}^{\infty} |z^i|$$
$$\leqslant \sum_{i=n(n+1)/2}^{\infty} r^i, \quad \text{where} \quad |z| \leqslant r < 1.$$

Since $\sum_{i=n(n+1)/2}^{\infty} r^i \to 0$, as $n \to \infty$, $f_n(z)$ converges uniformly to $E(z)$, as $n \to \infty$ on every closed circle, $|z| \leqslant r < 1$.

The singularities of a function can be deciphered from its Padé approximants [4, p.48]. We see that $E(z) \to \infty$ as $z \to 1$ along the real axis so that $z = 1$ is a singular point of $E(z)$. The poles of $f_n(z)$ are the nth roots of unity and they lie on the circle $|z| = 1$; that is, all points of the form $e^{(i2\pi k)/n}$, where k and n are integers, are singular points of $f_n(z)$. When n is very large, these points become dense on the unit circle so that every neighbourhood of any other point on the unit circle must contain at least one of these nth roots of unity. Hence no point on $|z| = 1$ is a regular point. That is, $|z| = 1$ is the natural boundary of $E(z)$.

Now let us see how the convergents of (3.11) also support the above fact. The relevant convergents are

$$\frac{1+z}{1}, \quad \frac{1+2z+z^2}{1+z-z^3}, \quad \frac{1+2z+2z^2+2z^3-z^5}{1+z+z^2-z^4-z^5},$$
$$\frac{1+2z+2z^2+3z^3+2z^4-z^7-z^8}{1+z+z^2+z^3-z^5-z^6-z^7},$$
$$\frac{1+2z+2z^2+3z^3+3z^4+2z^5+z^6-z^8-2z^9-z^{12}}{1+z+z^2+z^3+z^4-z^6-z^7-z^8-z^9}, \ldots.$$

Let $f_n(z) = P_n(z)/Q_n(z)$ be the nth convergent of (3.11). By making use of the induction method and with the help of (1.5), we obtain

$$Q_n(z) = 1 + z + z^2 + \cdots + z^{n-1} - z^{n+1} - z^{n+2} - \cdots - z^{2n-1}$$
$$= \frac{1-z^n}{1-z} - z^{n+1}\frac{(1-z^{n-1})}{1-z}. \tag{4.5}$$

From power series expansions, it follows that

$$f_2(z)=1+z+z^3+z^6+O(z^7),$$
$$f_3(z)=1+z+z^3+z^6+z^{10}+O(z^{12}),$$
$$f_4(z)=1+z+z^3+z^6+z^{10}+z^{15}+O(z^{18}),$$
$$\vdots$$

In general,

$$f_n(z)=1+z+z^3+z^6+\cdots+z^{(n+1)(n+2)/2}+O(z^{n(n+5)/2}),$$

which can be proved by the induction method. Therefore,

$$E(z)-f_n(z)=O(z^{n(n+5)/2}). \tag{4.6}$$

Also

$$f_2-f_1=\frac{z^3+z^4}{Q_1Q_2},\qquad f_3-f_2=\frac{z^7+z^8}{Q_2Q_3},\qquad f_4-f_3=\frac{z^{12}+z^{13}}{Q_3Q_4},$$
$$\vdots \tag{4.7}$$
$$f_n-f_{n-1}=\frac{z^{(n-1)(n+4)/2}+z^{(n^2+3n-2)/2}}{Q_{n-1}Q_n}.$$

In proving the uniform convergence of $f_n(z)$, we may assume that for each $0<r<1$, there is an integer $m(r)$ such that all zeros z of Q_n satisfy $|z|\geqslant r$. If not, for each integer n there exists an integer $m_n\geqslant n$ and a zero z_n of Q_{m_n} such that $|z_n|<r$. Then

$$\frac{1-z_n^{m_n}}{1-z_n}-z_n^{m_n+1}\frac{\left(1-z_n^{m_n-1}\right)}{1-z_n}=0,$$

that is, $1-z_n^{m_n}=z_n^{m_n+1}(1-z_n^{m_n-1})$. Therefore,

$$0<r\leqslant|1-z_n^{m_n}|=|z_n^{m_n+1}||1-z_n^{m_n-1}|\leqslant 2r^{m_n+1}\to 0,\quad \text{as } n\to\infty.$$

As this is impossible, our assumption is true. Also, for $|z|<r$,

$$\left|Q_n-\frac{1}{1-z}\right|=\left|\frac{1-z^n}{1-z}-\frac{z^{n+1}(1-z^{n-1})}{1-z}-\frac{1}{1-z}\right|=\left|\frac{1}{1-z}\right|\left|-z^n-z^{n+1}(1-z^{n-1})\right|$$
$$=\left|\frac{1}{1-z}\right||z^n||1+z-z^n|\leqslant\frac{1}{1-r}r^n(1+r+r^n)$$
$$\leqslant\frac{3r^n}{1-r}\to 0,\quad \text{as } n\to\infty.$$

Hence $Q_n(z)$ converges uniformly to $1/(1-z)$ on every closed circular disc with radius <1. By [17, Theorem 10.12, p.235] and by our assumption, $1/Q_n(z)$ converges uniformly to $1-z$ on every circular disk with radius <1. That is, for every $\epsilon>0$, there is an integer $n(r)$ such that

$$\left|\frac{1}{Q_n(z)}-(1-z)\right|<\epsilon,\quad \text{whenever } |z|<r \text{ and } n\geqslant n(r).$$

In particular,

$$\left|\frac{1}{Q_n(z)}\right| \leqslant \left|\frac{1}{Q_n(z)}-(1-z)\right|+|1-z| \leqslant \epsilon+2.$$

If we take $\epsilon=1$, say, for every $0<r<1$, there is an integer $n(r)$ such that $|1/Q_n(z)| \leqslant 3$, whenever $|z| \leqslant r$ and $n \geqslant n(r)$. Fix $0<r<1$ and n_r, as defined above. Then for $n \geqslant n_r+5$ we have

$$|f_n(z)-f_{n-1}(z)| \leqslant 18r^{(n-1)(n+4)/2} \leqslant 18r^n.$$

Therefore,

$$\left|\sum_{i=n}^{\infty}(f_i(z)-f_{i-1}(z))\right| \leqslant \sum_{i=n}^{\infty}|f_i(z)-f_{i-1}(z)| \leqslant 18\sum_{i=n}^{\infty} r^i \to 0, \quad \text{as } n \to \infty.$$

Thus $\sum_{i=n}^{\infty}(f_i(z)-f_{i-1}(z))$ converges uniformly on every closed circular disc with radius <1, that is,

$$f_m(z)=f_1(z)+\sum_{i=2}^{m}(f_i(z)-f_{i-1}(z))$$

converges uniformly on every closed circular disc with radius <1. Since the mth convergent $f_m(z)$ agrees with $E(z)$ for the first $\frac{1}{2}m(m+5)$ terms, $f_m(z)$ converges uniformly to $E(z)$ on every closed circular disc with radius <1.

Now let us look at the poles of the convergents of (3.11). They are given in Table 1. The poles are all located closely near the circumference of the unit circle $|z|=1$ (see Fig. 2). We notice that each convergent has a real positive pole which lies outside the circle. We also observe that exactly half of them lie outside and the other half inside the unit circle. When the order of the convergents increases, not only the poles move towards the circumference of the unit circle but also they become dense near to it. This means that $|z|=1$ is the natural boundary of $E(z)$. This conclusion is confirmed by the blocks in the Padé table of $E(z)$ to which now we turn.

5. Block structure of convergents in the Padé table

We now consider the particular aspect of blocks formation by the convergents of (2.1). The Padé table of the $E(z)$-series turns out to be nonnormal. The detailed structure of the nonnormal Padé table has already been studied by Padé, who proved the following very useful theorem [20, p.394].

Theorem. *Let $R(z)=P_{m,n}(z)/Q_{m,n}(z)$ be an $[m/n]$ Padé approximant of $f(z)$, where $P_{m,n}(z)$ and $Q_{m,n}(z)$ are exactly $a_0+a_1z+\cdots+a_pz^p$ and $b_0+b_1z+\cdots+b_qz^q$, respectively, such that they have no zeros in common and*

$$f(z)Q_{m,n}(z)-P_{m,n}(z)=O(z^{p+q+r+1}), \quad r \geqslant 0. \tag{5.1}$$

Then

$$[p+i/q+j]_f(z)=R(z), \quad i, j=0, 1, \ldots, r, \tag{5.2}$$

and no other entry of the Padé table of $f(z)$ is identical with R.

Table 1
Poles of the first 8 convergents of (3.11); results are rounded off to 5 significant digits

Order	$[m/n]$	Real part	Imaginary part	Modulus r
2	[2/3]	1.32471	0.0	1.32471
		−0.66235	±0.56227	0.86883
3	[5/5]	1.12373	0.0	1.12373
		0.17770	±0.85267	0.86975
		−0.94417	±0.55646	1.09595
4	[8/7]	1.06630	0.0	1.06[illegible]
		−0.88895	±0.35405	0.95680
		−0.36248	±1.08513	1.14407
		0.21828	±0.85727	0.88462
5	[12/9]	1.04152	0.0	1.04152
		0.42699	±0.79868	0.90560
		−0.57613	±0.72974	0.92975
		−0.98135	±0.32086	1.03247
		0.10973	±1.12180	1.12715
6	[17/11]	1.02850	0.0	1.02850
		−0.94720	±0.24937	0.97948
		−0.28875	±0.88161	0.92769
		0.56293	±0.72988	0.92176
		0.39630	±1.03193	1.10541
		−0.73753	±0.76832	1.06502
7	[23/13]	1.02080	0.0	1.02080
		−0.05759	±0.93031	0.93209
		−0.75979	±0.58868	0.95932
		0.65573	±0.66516	0.93403
		0.57135	±0.92528	1.08746
		−0.42943	±0.98250	1.07225
		−0.99067	±0.22680	1.01630
8	[30/15]	1.01585	0.0	1.01585
		0.12308	±0.92985	0.93796
		−0.54994	±0.77790	0.95266
		−0.85701	±0.58334	1.03670
		0.72168	±0.30788	0.94358
		−0.15947	±1.05808	1.07003
		0.68327	±0.82801	1.07352
		−0.96953	±0.19094	0.98815

Though $P_{m,n}(z)/Q_{m,n}(z)$ is a unique Padé fraction, $P_{m,n}(z)$ and $Q_{m,n}(z)$ are, in general, not unique. That is, if s is a fixed integer, $0 \leqslant s \leqslant r$, then all Padé approximants, where

$$m = p + s, \qquad n = q + s,\ q + s + 1, \ldots, q + r$$

or

$$m = p + s,\ p + s + 1, \ldots, p + r, \qquad n = q + s$$

will have the form

$$P_{m,n}(z) = \left(a_0 + a_1 z + \cdots + a_p z^p\right) G_s(z)$$

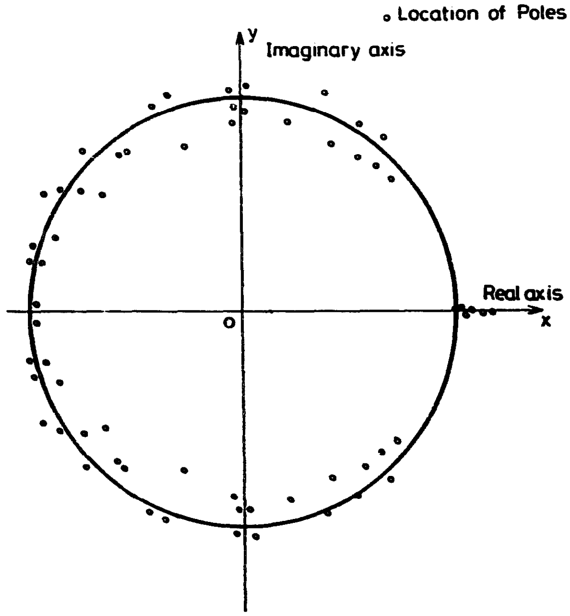

Fig. 2. The unit circle $|z| = 1$ with poles of the convergents of (3.1).

and

$$Q_{m,n}(z) = (b_0 + b_1 z + \cdots + b_q z^q) G_s(z),$$

$G_s(z)$ being an appropriate polynomial of degree s. The polynomial $G_s(z)$ is arbitrary, when $m + n \leqslant p + q + r$ and it is $z^t G_{s-t}(z)$, when $m + n = p + q + r + t$.

The above theorem shows that identical Padé approximants can occur only in square blocks of the Padé table. The block (5.2) consisting of $(r+1)^2$ elements is called a *block of order r*. This type of square block structure is seen in other Padé tables too [1]. Since the convergents of C-fraction are irreducible, we can very well apply the above theorem here. It is sufficient to find out the integer r in (5.1) for deciding the order of a block formed by a particular Padé approximant. The integer r is conveniently determined by comparing the power series expansion of $P_{m,n}(z)/Q_{m,n}(z)$ with the original series.

We present below a fresh working method for the expansion of a rational function into a power series. The expansion of $R(z) = P_{m,n}(z)/Q_{m,n}(z)$ in a series of positive powers of z is carried out as follows. Let

$$R(z) = \frac{a_{10} + a_{11} z + \cdots + a_{1p} z^p}{a_{00} + a_{01} z + \cdots + a_{0q} z^q}, \tag{5.3}$$

and also

$$R(z) = \frac{1}{a_{00}} (a_{10} + a_{20} z + a_{30} z^2 + \cdots), \tag{5.4}$$

where the coefficients in (5.4) can be expressed in terms of bigradients of (5.3) [12]. For example, the coefficient of z^{n-1} is given by

$$a_{n0} = \frac{1}{(a_{00})^{n-1}} \begin{vmatrix} a_{00} & a_{01} & \cdots & a_{0,n-1} \\ 0 & a_{00} & \cdots & a_{0,n-2} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{01} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \end{vmatrix}. \tag{5.5}$$

In the process of evaluating the bigradient (5.5), not only the coefficient a_{n0} but also all the coefficients $a_{20}, a_{30}, \ldots, a_{n0}$ are obtained directly. For this, we rewrite (5.5) as

$$a_{n0} = \begin{vmatrix} 1 & a_{01}/a_{00} & \cdots & a_{0,n-1}/a_{00} \\ 0 & 1 & \cdots & a_{0,n-2}/a_{00} \\ \vdots & & & \\ 0 & 0 & \cdots & a_{0,1}/a_{00} \\ a_{10} & a_{11} & \cdots & a_{1,n-1} \end{vmatrix}. \tag{5.6}$$

In evaluating the determinant a_{n0}, its order is reduced successively by expanding it in terms of the first column after going through all appropriate elementary transformations. At each stage of the order reduction, the first elements of the last row of these reduced-order determinants are noted and they are the coefficients $a_{20}, a_{30}, \ldots, a_{n0}$ of (5.4). The computation involved in this process can conveniently be displayed in the form of an array:

$$\begin{array}{ccccc} a_{00} & a_{01} & a_{02} & a_{03} & \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & \cdots \\ a_{20} & a_{21} & a_{22} & a_{23} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

The first two rows of the above array are formed from the coefficients in (5.3) and the elements of the third, fourth and subsequent rows can be evaluated from the recursive relation

$$a_{i,j} = a_{i-1,j+1} - a_{i-1,0}a_{0,j+1}/a_{00}, \quad i \geqslant 2, \ j \geqslant 0. \tag{5.7}$$

If we have the ratio of two infinite series in the place of a proper rational function, the procedure is slightly different; that is, to get the first N terms of the required series we have to take the first N terms in each of the infinite series and perform the above computation. It is enough to evaluate the triangular array in order to get the required number of coefficients.

Since we now know the power series expansion of $R(z)$, the relation (5.1) can be rewritten in an equivalent form:

$$f(z) - R(z) = O(z^{p+q+r+1}), \quad r \geqslant 0. \tag{5.8}$$

Let us identify each convergent of a C-criterion with $R(z)$ of (5.8). Then $R(z)$ has a contact of order $p+q+r$ with $f(z)$. We shall denote this property by the notation

$$C[R(z)] = p+q+r. \tag{5.9}$$

The integer r in (5.8) plays an important role in investigating the block structure of the Padé table. For every convergent of a C-fraction the integer r exists. If $r = 0$, $R(z)$ is called a *normal*

Padé approximant; that is, it forms a block of order zero. If $r > 0$, then $R(z)$ is a nonnormal Padé approximant whose block order is r. As we have already seen, some of the convergents of (2.1) are not Padé fractions. Hence it is necessary to relax the condition $r \geqslant 0$ in (5.8); that is, r may be negative. If $r < 0$, then $R(z)$ finds no place in the Padé table, which means that it forms a block of "negative order". Let us represent the integer r symbolically by

$$\mathrm{Ord}[R(z)] = C[R(z)] - (p+q). \tag{5.10}$$

A nonnormal Padé approximant can in fact be of two types. One is termed *surplus* $(r > 0)$ and the other *partial* $(r < 0)$. The non-Padé convergent of a C-fraction may be called *Partial Padé Approximant* (PPA). The terminology of PPA is not new [13]. It is currently being used in the theory of control systems.

We shall look at the blocks formed by the convergents of (2.1) in the Padé table of $E(z)$. By comparing (1.3) with (2.9) we find that the integer r_n exists such that

$$p_n + q_n + r_n + 1 = \alpha_1 + \alpha_2 + \cdots + \alpha_{n+1}, \quad r_n \geqslant 0, \; n = 1, 2, \ldots . \tag{5.11}$$

Let B_n denote the block of the nth convergent $P_n(z)/Q_n(z)$. Since $P_n(z)/Q_n(z)$ is irreducible, its order is r_n. Then it follows from the Padé theorem that $P_n(z)/Q_n(z)$ occupies all the cells of the block B_n and occurs nowhere else in the table. The positions of the convergents in the Padé table and the blocks formed out of them are shown in Fig. 1. The convergents arrange themselves to form a zig-zag path. For $n = 1$, we have $p_1 = 0$, $q_1 = 0$ so that $r_1 = \alpha_2 - 1 = 0$; that is, the order of the block B_1 is zero and therefore it is the cell (0, 0). If $n = 2$, $p_2 = 0$, $q_2 = 1$ and $r_2 = \alpha_2 + \alpha_3 - 2 = 0$. Therefore, the block B_2 is only the cell (0, 1). Since $p_3 = 1$, $q_3 = 0$, $r_3 = \alpha_1 + \cdots + \alpha_4 - 2 = 1$, the block B_3 consists of 2×2 cells and they are (1, 0), (1, 1), (2, 0) and (2, 1). Proceeding along similar lines we find that the orders of blocks, for example, B_4, B_5 and B_6 are found to be 0, 0 and 1, respectively. In view of (5.8) we see that the 1st, 2nd, 4th, 5th order convergents of (2.1) are normal Padé approximants, while the remaining convergents are all nonnormal. Their Padé realization and block structures are depicted in Table 2. For convergents of order $3n + 2$, $n \geqslant 2$, (5.11) is

$$p_{3n+2} + q_{3n+2} + r_{3n+2} + 1 = \alpha_1 + \cdots + \alpha_{3n+2},$$

Table 2
Results of the convergents of (2.1)

Order	$[m/n]$	$\mathrm{Ord}[R(z)] = r$	Type of approximant
1	[0/0]	0	Normal
2	[0/1]	0	Normal
4	[1/2]	0	Normal
5	[2/2]	0	Normal
3	[1/0]	1	Surplus
6	[2/3]	1	Surplus
$3n+2$, $n = 2, 3, \ldots$	$[\frac{1}{2}n^2 + n + 2)/(n+1)]$	$n-1$	Surplus
$3(n+1)$, $n = 2, 3, \ldots$	$[\frac{1}{2}(n^2 + n + 4)/(2n+1)]$	$n-1$	Surplus
$3n+1$, $n = 2, 3, \ldots$	Almost Padé fraction	-1	Partial

that is,

$$\tfrac{1}{2}(n^2 + 3n + 6) + r_{3n+2} = \tfrac{1}{2}(n+1)(n+4),$$

that is,

$$r_{3n+2} = n - 1. \tag{5.12}$$

Similarly for the $3(n+1)$th order convergents, $n \geqslant 2$, (5.11) yields

$$r_{3(n+1)} = n - 1. \tag{5.13}$$

Equations (5.12) and (5.13) show that the order of the successive blocks B_{3n+2} and $B_{3(n+1)}$, $n \geqslant 2$, are equal. Since

$$p_{3(n+1)} - p_{3n+2} = 1, \quad n = 2, 3, \ldots, \tag{5.14}$$

the position of one block is just one cell lower than that of the other. Also

$$q_{3(n+1)} - q_{3n+2} = n, \quad n = 2, 3, \ldots . \tag{5.15}$$

This implies that the common line segment of the blocks B_{3n+2} and $B_{3(n+1)}$ is progressively increasing. Further, the geometric configuration of the left-side blocks is represented by

$$\left.\begin{aligned} p_{3n+5} - p_{3n+2} &= r_{3n+2} + 2, \\ q_{3n+5} - q_{3n+2} &= 1, \end{aligned}\right\} \quad n = 1, 2, \ldots . \tag{5.16}$$

The corresponding relations for the right-side blocks are

$$\left.\begin{aligned} p_{3(n+2)} - p_{3(n+1)} &= r_{3(n+1)} + 2, \\ q_{3(n+2)} - q_{3(n+1)} &= 2, \end{aligned}\right\} \quad n = 1, 2, \ldots . \tag{5.17}$$

It is observed clearly that the order of blocks are increasing regularly. In this context, it is worthwhile to recall the prophetic words of Wall [20]: "We might be permitted to conjecture that the nature of the blocks in the table have a bearing upon the nature of the function represented by the power series. Perhaps any power series whose Padé table contains a sequence of blocks whose orders increase sufficiently rapidly represents a function with a natural boundary".

6. Concluding remarks

A study of the C-fraction expansion of the Eisenstein–Ramanujan continued fraction brings out some new aspects such as partial Padé approximants and two-third part contractions. An explicit method of expansion of rational functions into power series is exploited in order to determine the exact contact properties of the convergents. The convergents of C-fraction expansion of $E(z)$ reveal that the unit circle is the natural boundary for the series. An important observation is that the blocks of convergents lie not around the main diagonal of the Padé table but are far away. Perhaps, this is due to the presence of non-Padé convergents, which occur periodically.

References

[1] P. Achuthan and S. Ponnuswamy, On general two point continued fraction expansions and Padé tables, *J. Approx. Theory* **64** (1991) 291–314.
[2] G.E. Andrews, An introduction to Ramanujan's 'lost' notebook, *Amer. Math. Monthly* **86** (1979) 89–108.
[3] G.A. Baker Jr, *Essentials of Padé Approximants* (Academic Press, New York, 1975).
[4] G.A. Baker Jr and P.R. Graves-Morris, *Padé Approximants, Part I: Basic Theory*, Encyclopedia Math. Appl. **13** (Addison-Wesley, Reading, MA, 1981).
[5] S. Bhargava and C. Adiga, On some continued fraction identities of Srinivasa Ramanujan, *Proc. Amer. Math. Soc.* **92** (1984) 13–18.
[6] A. Bultheel, Recursive algorithms for non-normal Padé tables, *SIAM J. Appl. Math.* **39** (1980) 106–118.
[7] A. Cuyt and L. Wuytack, *Nonlinear Methods in Numerical Analysis* (North-Holland, Amsterdam, 1987).
[8] G. Eisenstein, Transformations remarquables de quelques series, *J. Reine Angew. Math.* **27** (1844) 193–197; also in: *Mathematische Werke, Band 1* (Chelsea, New York, 1975) 35–39.
[9] E. Frank, Corresponding type continued fractions, *Amer. J. Math.* **68** (1946) 89–108.
[10] W.B. Gragg, The Padé table and its relation to certain algorithms of numerical analysis, *SIAM Rev.* **14** (1972) 1–62.
[11] M.D. Hirschhorn, A continued fraction of Ramanujan, *J. Austral. Math. Soc. Ser. A* **29** (1980) 80–86.
[12] A.S. Householder, *The Numerical Treatment of a Single Nonlinear Equation* (McGraw-Hill, New York, 1970).
[13] M.F. Hutton and B. Friedland, Routh approximations for reduced order of linear time-invariant systems, *IEEE Trans. Automat. Control* **AC-20** (1975) 329–337.
[14] W.B. Jones and W.J. Thron, *Continued Fractions, Analytic Theory and Applications*, Encyclopedia Math. Appl. **11** (Addison-Wesley, Reading, MA, 1980).
[15] W. Leighton and W.T. Scott, A general continued fraction expansion, *Bull. Amer. Math. Soc.* **45** (1939) 596–605.
[16] S. Ramanujan, *Notebooks* (Tata Institute of Fundamental Research, Bombay, 1957); (Springer, Berlin, 1987).
[17] W. Rudin, *Functional Analysis* (Tata McGraw-Hill, New Delhi, 1974).
[18] W.T. Scott and H.S. Wall, Continued fraction expansions for arbitrary power series, *Ann. of Math. (2)* **41** (1940) 328–349.
[19] E.C. Titchmarch, *The Theory of Functions* (Clarendon Press, Oxford, 1932).
[20] H.S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, New York, 1948).